Countable stable unidimensional theories are superstable

by Udi Hrushovski

L is countable, T is stable, and we work in C^{eq}. For convenience assume $acl(\emptyset)=dcl(\emptyset)$.

Let $S_x^* = S_x^*(\emptyset)$ be the set of x-types based on $\emptyset$, thought of as definition-schemes; thus for q in S_x^* and a formula $R(x;\bar{y})$ in L, write simply qR for the q-definition of R. Then it becomes natural to endow S_x^* with the following topology: a basic open set has the form $\{q: qR_1=S_1,\ldots,qR_n=S_n\}$ for given $R_1(x,\bar{y}_1), S_1(\bar{y}_1), \ldots, R_n(x,\bar{y}_n), S_n(\bar{y}_n)$. (By Sh , restricting to n=1 above would make no difference.)

The following are immediate:

1) S_x^* is a Polish space.

2) Let "f(x)=y" be a definable function. Then f induces a natural map $S_x^* \to S_y^*$; this map is trivially continuous, and is an open map if T has the nfcp. (Prove this first for $S_x^*(M)$.)

3) The maps $S_x^* \times S_y^* \xrightarrow{\otimes} S_{xy}^*$ are all continuous. $(p,q \mapsto p \otimes q)$.

From now on fix a type p in S_y^*. Let f, η range over all definable fn's f (from the sort of x to any other sort) and over formulas $\eta(z,\bar{u},\bar{y})$ s.t. $\models \forall \bar{u}\bar{y} \exists^{<\aleph_0} z\, \eta(z,\bar{u},\bar{y})$. Write: "$f(q) \overset{?}{<} p$" if for some $\bar{b}$, there exist $a \models q \restriction \bar{b}$ and $\bar{d}$ s.t. each $d_i \models p \restriction \bar{b}$, and s.t. $\models \eta(f(a),\bar{b},\bar{d})$; and let the notation imply also that f(a) above is not in $acl(\bar{b})$. Then by Sh V,4. we have: $q \not\perp p$ iff for some f and η, $f(q) \overset{?}{<} p$. Hence

<u>Claim</u> If no $q \in S_x^*$ is orthogonal to p, then for some η, f the set $W_{\eta,f} = \{q: f(q) \overset{?}{<} p\}$ has non-empty interior. $\#$

<u>Proof</u> This follows from Baire category provided that each $W_{\eta,f}$ is closed. To see this, let $q \in cl(W_{\eta,f})$. Then (by considering sufficiently close q' 's) for every finite set $R_1(\bar{u},y), \ldots, R_k(\bar{u},y)$, we have:

$$q(\ (\exists \bar{y})(\bigwedge_j \bigwedge_{i \le k} R_i(\bar{u},y_j)=pR_i(\bar{u})\ \ \&\ \ \eta(f(x),\bar{u},\bar{y})\ )\ )\ \text{ is consistent.}$$

By compactness, there exists $\bar{b}$ realizing all of the above formulas; and then it is easu to find a and (using compactness again) $\bar{d}$ showing that $q \in W_{\eta,f}$. So $W_{\eta,f}$ is closed. $\square$

Now assume that T is uni-dimensional, and WLOG that $dcl(\emptyset)$ is the universe of a model M, and that the type p above is minimal. By the claim, some $W_{\eta,f}$ has non-empty interior. Now T has the nfcp, so point (2) above applies, showing: $\{stp(f(a)/M): stp(a/M) \in W_{\eta,f}\}$ also has non-empty interior. Thus for some $r(z) \in S_z^*$ and some formula $R(z,\bar{v})$, we have: $r' \in S_z^{**}$, $r'R = rR$ implies $r \not\perp p$. In particular:

(#) $r'R = rR$ implies $U(r') < \infty$.

Let $r^* \in S(\mathfrak{C})$ be a minimal type extending $r \restriction M$, and pick a formula S^* in r^* such that $R\text{-}M(\ S^*(z)\ ;\ R(z;\bar{v})\ ) = R\text{-}M(\ q^*\ ;\ R(z;\bar{v})\ ) = 1,1$. Then if $r'(z)$ is a global type with $S \in r'$, then either r' is algebraic, or:

$R(z,\bar{a}) \in r'$ iff $R(z,\bar{a}) \in r^*$ (for $\bar{a} \in \mathfrak{C}$). In particular, this holds for $\bar{a}$ from M; so letting $r'' \stackrel{=}{_{df}} r' \restriction M$, we have: $R(z,m) \in r''$ iff $R(z,m) \in r$. Hence $r''R = rR$, so by (#) $U(r'')$ exists. But r' extends r'', so $U(r')$ exists also.

We have a formula S^* now such that every type inside S^* has U-rank. Since T is uni-dimensional, the universe of C is "analyzable" in terms of U^C in finitely many steps, so every type has U-rank. (Cf. Ba and Bu or H or Sh V.7)

Hence T is superstable.

Remark If T is stable, uni-dimensional, but not necessarily countable, one may show: $\kappa_r(T) \le card(T)$.

Ba	Baldwin,	
Bu	Buechler, S.	A Theorem on Uni-dimensional Theories
H	Hrushovski, E	Kueker's Conjecture for Stable Theories
Sh	Shelah, S.	Classification Theory

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